PHYS2217 discussion

Ameen Ismail

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1 Infinite wire

Consider an infinitely long wire carrying a steady current I in the $+\hat{\mathbf{z}}$ direction. Use the Biot-Savart law to find the magnetic field $\vec{\mathbf{B}}$ at a point a distance s from the wire.

Hints: your result should agree with that obtained via Ampere's law:

$$B = \frac{\mu_0 I}{2\pi s}. \tag{1}$$

And recall the Biot-Savart law for a line current is

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int \frac{\mathrm{d}\vec{\ell} \times \hat{\mathbf{r}}}{r^2} \tag{2}$$

where the line integral is taken along the current and $\vec{\mathbf{r}}$ is the vector from a point on the wire to the point of interest.

2 Sheet of current

Consider an infinite sheet of current lying in the xy-plane, carrying a surface current K in the $+\hat{\mathbf{x}}$ direction. Use Ampere's law to find the magnetic field $\vec{\mathbf{B}}$ above and below the plane.

Hints: The magnetic field can only point in the $\hat{\mathbf{y}}$-direction (why?). Ampere's law in integral form is

$$\int \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{enc} \tag{3}$$

where the line integral is taken over a closed loop and I_{enc} is the total current that loop encloses.

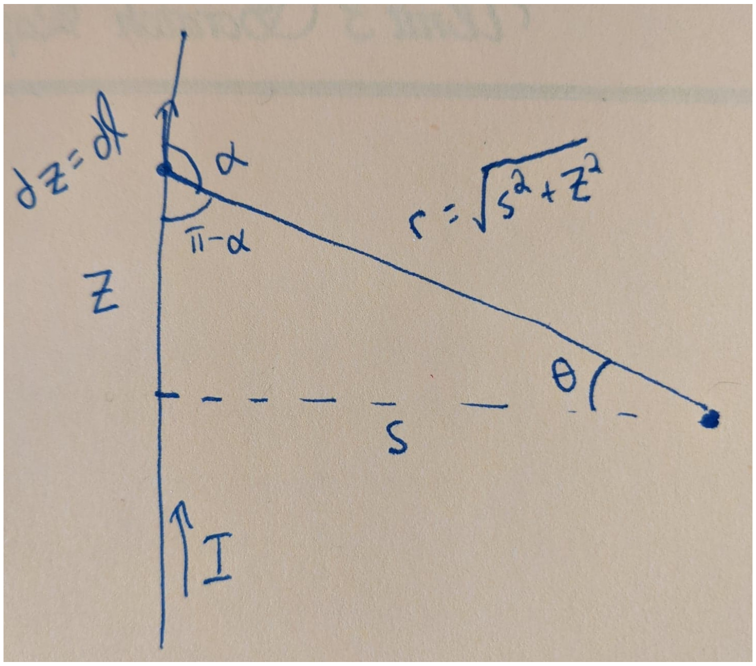

Figure 1: Diagram for infinite wire, hand-drawn.

Solution: infinite wire

We apply the Biot-Savart law directly:

$$B = \frac{\mu_0 I}{4\pi} \int \frac{\mathrm{d}\vec{\ell} \times \hat{\mathbf{r}}}{r^2}. \tag{4}$$

Consulting the diagram, we see $\mathrm{d}\vec{\ell} \times \hat{\mathbf{r}} = \mathrm{d}z \sin\alpha$. Now we note

$$\sin\alpha = \sin(\pi - \alpha) = \cos\theta = \frac{s}{r}. \tag{5}$$

Plugging this into the Biot-Savart law, we have

$$B = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{+\infty} \frac{s\,\mathrm{d}z}{r^3} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{+\infty} \frac{s\,\mathrm{d}z}{(s^2 + z^2)^{3/2}}. \tag{6}$$

This integral should look familiar. We can evaluate it using trig substitution, changing variables from z to θ using $z = s\tan\theta$. The integral evaluates to just $2/s$, giving us a final answer of

$$B = \frac{\mu_0 I}{2\pi s} \tag{7}$$

as expected.

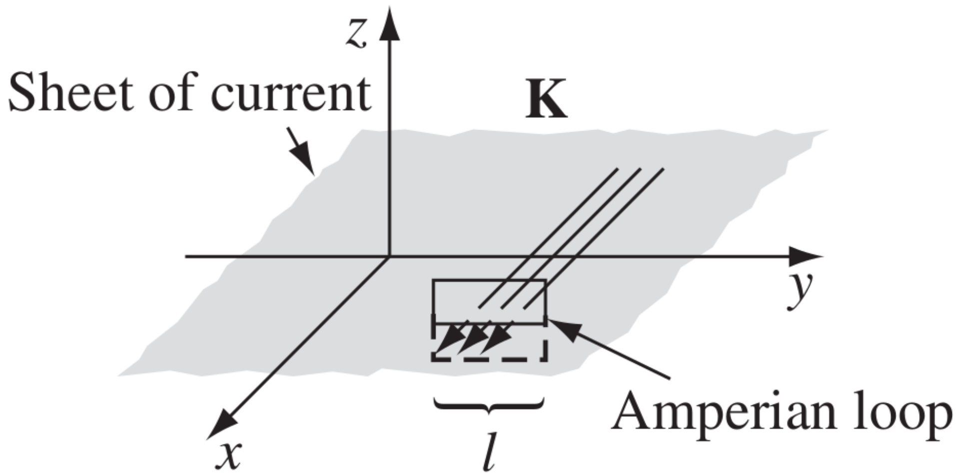

Figure 2: Diagram for current sheet, shamelessly stolen from Griffiths §5.3.3.

Solution: sheet of current

This problem is similar to finding the electric field from an infinite sheet of charge. The magnetic field is constant above and below the plane, irrespective of z. The right hand rule shows that $\vec{\mathbf{B}}$ must point in the $-\hat{\mathbf{y}}$ direction above the plane and in the $+\hat{\mathbf{y}}$ direction below. By symmetry, the magnitude of the magnetic field must be the same above and below the sheet.

Now let us draw a rectangular Amperian loop lying in the yz-plane, with one end above the sheet and the other end below. Suppose its length in the y-direction is L. (Its height in the z direction does not matter.) We want to compute the line integral

$$\int_{\text{loop}} \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\ell}. \tag{8}$$

The sides of the rectangle lying in the z-direction contribute nothing to the line integral, since $\mathrm{d}\vec{\ell}$ and $\vec{\mathbf{B}}$ are orthogonal. The integral over the top and bottom of the loop each contribute an amount BL, and so we have

$$\int_{\text{loop}} \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\ell} = 2BL. \tag{9}$$

The enclosed current is just the the surface current multiplied by the length of the loop $I_{enc} = KL$. Plugging this into Ampere's law, we have

$$2BL = \mu_0 KL \tag{10}$$

and so $B = \mu_0 K/2$. We conclude that

$$\vec{\mathbf{B}} = \pm\mu_0 \frac{K}{2}\hat{\mathbf{y}} \tag{11}$$

positive for $z < 0$ and negative for $z > 0$.