# PHYS2217 discussion

Ameen Ismail

27 April 2020

## 1 Maxwell's equations and Lorentz invariance

In this problem we will write Maxwell's equations using the field strength tensor $F_{\mu\nu}$ to show that they are Lorentz invariant. Show that

$$\frac{\partial}{\partial x^\mu} F^{\mu\nu} = 0 \tag{1}$$

is equivalent to Gauss' law and Ampère's law ("inhomogeneous Maxwell's equations") in vacuum. Note the implicit summation over the index $\mu$. Argue that this equation is Lorentz invariant.

Next, show the other two ("homogeneous") Maxwell equations are given by

$$\frac{\partial}{\partial x^\mu} \tilde{F}^{\mu\nu} = 0 \tag{2}$$

where $\tilde{F}^{\mu\nu}$ is the electromagnetic *dual tensor*. It is defined by $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$, where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric symbol.

Equation 2 is in fact equivalent to the condition

$$\frac{\partial}{\partial x^\mu} F_{\nu\rho} + \frac{\partial}{\partial x^\nu} F_{\rho\mu} + \frac{\partial}{\partial x^\rho} F_{\mu\nu} = 0 \tag{3}$$

(Bonus: prove they're equivalent.) Because $F_{\mu\nu}$ is antisymmetric, it can be written as

$$F_{\mu\nu} = \frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu \tag{4}$$

where $A_\mu = (\phi, \vec{\mathbf{A}})$ is the four-potential. By writing the field strength tensor in this way, show that the homogeneous Maxwell equations are trivially satisfied.

*Hints*: The field strength tensor, in Cartesian coordinates, is related to the electric and magnetic fields by

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{5}$$

The dual tensor $\tilde{F}^{\mu\nu}$ can be obtained from $F^{\mu\nu}$ by replacing $\vec{\mathbf{E}}$ with $\vec{\mathbf{B}}$ and $\vec{\mathbf{B}}$ with $-\vec{\mathbf{E}}$.

## Solution

Equation 1 is really four equations in one, one for each value of the index $\nu$. First take $\nu = 0$, giving the equation $\partial_\mu F^{\mu 0} = 0$. Since $F^{00} = 0$ and $F^{i0} = E_i$ (with $i = 1, 2, 3$), we then find

$$0 = \partial_x E_x + \partial_y E_y + \partial_z E_z = \boldsymbol{\nabla} \cdot \vec{\mathbf{E}} \tag{6}$$

which is Gauss' law in vacuum.

Next take $\nu = 1$: $\partial_\mu F^{\mu 1} = 0$. Again, directly subsituting the components of $F^{\mu 1}$ into this equation yields

$$0 = \partial_t E_x - \partial_y B_z + \partial_z B_y. \tag{7}$$

The $\nu = 2$ and $\nu = 3$ equations similarly give $0 = \partial_t E_y - \partial_z B_x + \partial_x B_z$ and $0 = \partial_t E_z - \partial_x B_y + \partial_y B_x$. We can neatly package these three equations into one as

$$0 = \frac{\partial \vec{\mathbf{E}}}{\partial t} - \boldsymbol{\nabla} \times \vec{\mathbf{B}} \tag{8}$$

which is Ampère's law!

Now consider the homogeneous Maxwell equations. Equation 2 is the same as equation 1 with $F^{\mu\nu}$ replaced with the dual tensor $\tilde{F}^{\mu\nu}$. Explicitly, the components of the dual tensor are

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{9}$$

So we can be clever and just use our previous result, but replace $\vec{\mathbf{E}}$ with $\vec{\mathbf{B}}$ and $\vec{\mathbf{B}}$ with $-\vec{\mathbf{E}}$. The $\nu = 0$ equation then gives

$$\boldsymbol{\nabla} \cdot \vec{\mathbf{B}} = 0 \tag{10}$$

and the $\nu = 1, 2, 3$ equations gives

$$0 = \frac{\partial \vec{\mathbf{B}}}{\partial t} + \boldsymbol{\nabla} \times \vec{\mathbf{E}}. \tag{11}$$

These are, of course, Gauss' law for magnetism and Faraday's law.

Now we write the field strength tensor as $F_{\mu\nu} = \frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu$ and substitute this directly into the alternative formulation of the homogeneous Maxwell equations that I gave. We find:

$$\begin{aligned} 0 &= \frac{\partial}{\partial x^\mu} F_{\nu\rho} + \frac{\partial}{\partial x^\nu} F_{\rho\mu} + \frac{\partial}{\partial x^\rho} F_{\mu\nu} \\ &= \frac{\partial}{\partial x^\mu}\left(\frac{\partial}{\partial x^\nu} A_\rho - \frac{\partial}{\partial x^\rho} A_\nu\right) + \frac{\partial}{\partial x^\nu}\left(\frac{\partial}{\partial x^\rho} A_\mu - \frac{\partial}{\partial x^\mu} A_\rho\right) + \frac{\partial}{\partial x^\rho}\left(\frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu\right). \end{aligned} \tag{12}$$

This *must* vanish since partial derivatives commute. Being able to write $F_{\mu\nu}$ in this form follows merely from the fact that it is an antisymmetric tensor. So, we conclude that the homogeneous Maxwell equations are in fact a consequence of the antisymmetry of the field strength tensor.

Finally, let us return to the question of Lorentz invariance. The quantity $\partial_\mu F^{\mu\nu}$ behaves as a four-vector under Lorentz transformations. Maxwell's equations in vacuum (equation 1) state that this quantity vanishes. If a four-vector is zero in one reference frame, it must be zero in all reference frames. So, that equation is invariant under Lorentz transformations. Furthermore, $F_{\mu\nu}$ is antisymmetric in any reference frame, and so the homogeneous Maxwell equations will be satisfied in any frame as well.

By the way, you may be thinking that the nice cancellation that occurred when we wrote $F_{\mu\nu}$ in terms of derivatives of $A_\mu$ hints at some deeper mathematical structure. Indeed, it does! Electromagnetism can be formulated in the language of differential forms, where Maxwell's equations look even simpler than in our discussion today. You will probably encounter this in your classes in the next few years.